# Problem's set #4

THEORETISCHE QUANTENOPTIK UND QUANTEN INFORMATION
2014S PS 705854
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## The Born-Markov Master Equation

1. **Properties of master equations: miscellanea**. Consider a general master equation in the Lindblad form
$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[\hat{H}, \hat{\rho}] + \sum_\alpha \Gamma_\alpha \left( \hat{a}_\alpha \hat{\rho} \hat{a}_\alpha^\dagger - \frac{1}{2}[\hat{a}_\alpha^\dagger \hat{a}_\alpha, \hat{\rho}]_+ \right), \tag{1}$$
for a given set of jump operators $\hat{a}_\alpha$ and decoherence rates $\Gamma_\alpha > 0$.

   (a) Show that
$$\mathrm{tr}[\dot{\rho}] = 0. \tag{2}$$

   (b) Show that the time derivative of the purity of a density matrix is given by
$$\frac{d}{dt}\mathrm{tr}[\hat{\rho}^2] = 2\sum_\alpha \Gamma_\alpha \left( \mathrm{tr}[\hat{\rho}\hat{a}_\alpha^\dagger \hat{\rho} \hat{a}_\alpha] - \mathrm{tr}[\hat{\rho}^2 \hat{a}_\alpha^\dagger \hat{a}_\alpha] \right). \tag{3}$$
   For the particular case of a two-level system with spontaneous emission, namely
$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[\hat{H}, \hat{\rho}] + \Gamma \left( \hat{\sigma}^- \hat{\rho} \hat{\sigma}^+ - \frac{1}{2}[\hat{\sigma}^+ \hat{\sigma}^-, \hat{\rho}]_+ \right), \tag{4}$$
   show that then,
$$\frac{d}{dt}\mathrm{tr}[\hat{\rho}^2] = -2\Gamma \left[ \rho_{ee}(2\rho_{ee} - 1) + |\rho_{eg}|^2 \right]. \tag{5}$$

   (c) Using the identity
$$f(t) = f(0) + \int_0^t d\tau \dot{f}(\tau), \tag{6}$$
   show that the Laplace transform of $f(t)$ is given by
$$\mathcal{L}[f](s) \equiv \int_0^\infty e^{-st} f(t) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}[\dot{f}](s), \tag{7}$$
   and thus
$$\mathcal{L}[\dot{f}](s) = s\mathcal{L}[f](s) - f(0) \tag{8}$$
   *Hint*: Use that in the two-dimensional integral over $\tau < t$, one can interchange the order of integration via $\int_0^\infty dt \int_0^t d\tau = \int_0^\infty d\tau \int_\tau^\infty dt$.
   We will use this relation to solve the optical Bloch equations in the next exercise.

2. **Damped Rabi Oscillations**. Using the master equation of a driven two-level system in the rotating frame with the laser frequency:
$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[-\Delta|e\rangle\langle e| + \frac{\Omega}{2}(\hat{\sigma}^+ + \hat{\sigma}^-), \hat{\rho}] + \Gamma \left( \hat{\sigma}^- \hat{\rho} \hat{\sigma}^+ - \frac{1}{2}[\hat{\sigma}^+ \hat{\sigma}^-, \hat{\rho}]_+ \right), \tag{9}$$

(a) Show that

$$\partial_t \begin{pmatrix} \langle\hat{\sigma}^x\rangle \\ \langle\hat{\sigma}^y\rangle \\ \langle\hat{\sigma}^z\rangle \end{pmatrix} = \begin{pmatrix} -\Gamma/2 & \Delta & 0 \\ -\Delta & -\Gamma/2 & -\Omega \\ 0 & \Omega & -\Gamma \end{pmatrix} \begin{pmatrix} \langle\hat{\sigma}^x\rangle \\ \langle\hat{\sigma}^y\rangle \\ \langle\hat{\sigma}^z\rangle \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \Gamma \end{pmatrix}. \tag{10}$$

This equation can be written as $\partial_t\langle\vec{\sigma}\rangle = \mathbf{M}\langle\vec{\sigma}\rangle - \mathbf{R}$.

(b) Using Eq. (8), show that

$$\mathcal{L}[\langle\vec{\sigma}\rangle](s) = \frac{1}{s(s-\mathbf{M})}(s\langle\vec{\sigma}(0)\rangle - \mathbf{R}). \tag{11}$$

Thus, note that by performing the inverse Laplace transform we obtain the solution of $\langle\vec{\sigma}(t)\rangle$.

(c) Assuming $\delta = 0$, and $\rho(0) = |g\rangle\langle g|$, make a plot of $\langle\hat{\sigma}^z(t)\rangle$ as a function of $\Omega t$ (with values ranging from $\Omega t = 0$ to $\Omega t = 6\pi$) for $\Gamma = 0$, $\Gamma = \Omega/10$, $\Gamma = \Omega/2$, $\Gamma = \Omega$, $\Gamma = 5\Omega$.

3. **Coherent Population Trapping**. In this problem, one needs to recall the question 3 of Problem's Set # 3 for a three-level atom in a lambda scheme. Here we consider spontaneous emission from the state $|e\rangle$ to $|g_1\rangle$ with a rate $\Gamma_1$ and from the state $|e\rangle$ to $|g_2\rangle$ with a rate $\Gamma_1$, see Fig. 1).

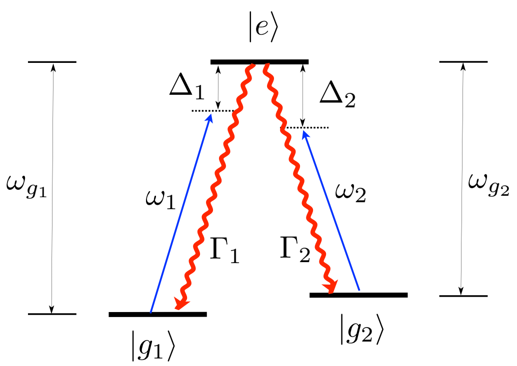


Figure 1: Lambda scheme.

The master equation of the system is given by

$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[\tilde{H}_p + \tilde{H}_I, \hat{\rho}] + \Gamma_1\mathcal{D}[\hat{\sigma}_1^-]\hat{\rho} + \Gamma_2\mathcal{D}[\hat{\sigma}_2^-]\hat{\rho}, \tag{12}$$

where the Hamiltonian is

$$\frac{\tilde{H}_p}{\hbar} = \Delta_1|g_1\rangle\langle g_1| + \Delta_2|g_2\rangle\langle g_2|, \tag{13}$$

$$\frac{\tilde{H}_I}{\hbar} = \frac{\Omega_1}{2}\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \frac{\Omega_2}{2}\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right), \tag{14}$$

and the dissipative terms are

$$\mathcal{D}[\hat{\sigma}_{1(2)}^-]\hat{\rho} = \hat{\sigma}_{1(2)}^-\hat{\rho}\hat{\sigma}_{1(2)}^+ - \frac{1}{2}\left(\hat{\sigma}_{1(2)}^+\hat{\sigma}_{1(2)}^-\hat{\rho} + \hat{\rho}\hat{\sigma}_{1(2)}^+\hat{\sigma}_{1(2)}^-\right). \tag{15}$$

We have defined $\hat{\sigma}_1^- = |g_1\rangle\langle e|$ and $\hat{\sigma}_2^- = |g_2\rangle\langle e|$. We have considered $\Omega_{1(2)}$ to be real and we have performed the RWA in the rotating frame defined by $\hat{U}(t) = \exp\left[-\mathrm{i}(\omega_1|g_1\rangle\langle g_1| + \omega_2|g_2\rangle\langle g_2|)t\right]$ (recall that $\Delta_{1(2)} = \omega_{1(2)} - \omega_{g_{1(2)}}$).

(a) Show that in the the dark-bright basis $\{|d\rangle, |b\rangle, |e\rangle\}$, where

$$\begin{aligned} |b\rangle &= \cos\theta|g_1\rangle + \sin\theta|g_2\rangle, \\ |d\rangle &= -\sin\theta|g_1\rangle + \cos\theta|g_2\rangle, \end{aligned} \tag{16}$$

with $\tan\theta = \Omega_2/\Omega_1$, the master equation can be written as

$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[\tilde{H}, \hat{\rho}] + \Gamma_b \mathcal{D}[\hat{\sigma}_b^-]\hat{\rho} + \Gamma_d \mathcal{D}[\hat{\sigma}_d^-]\hat{\rho} + (\Gamma_2 - \Gamma_1)\sin\theta\cos\theta\left(\hat{\sigma}_d^-\hat{\rho}\hat{\sigma}_b^+ + \hat{\sigma}_b^-\hat{\rho}\hat{\sigma}_d^+\right), \tag{17}$$

where

$$\frac{\tilde{H}}{\hbar} = \Delta_d|d\rangle\langle d| + \Delta_b|b\rangle\langle b| + \frac{\Omega_g}{2}\left(|d\rangle\langle b| + |b\rangle\langle d|\right) + \frac{\Omega_b}{2}\left(\hat{\sigma}_b^- + \hat{\sigma}_b^+\right), \tag{18}$$

and

$$\begin{aligned} \Delta_b &= \Delta_1\cos^2\theta + \Delta_2\sin^2\theta, \\ \Delta_d &= \Delta_1\sin^2\theta + \Delta_2\cos^2\theta, \\ \Omega_g &= 2(\Delta_2 - \Delta_1)\sin\theta\cos\theta, \\ \Omega_b &= \Omega_1\cos\theta + \Omega_2\sin\theta = \sqrt{\Omega_1^2 + \Omega_2^2}, \\ \Gamma_b &= \Gamma_1\cos^2\theta + \Gamma_2\sin^2\theta, \\ \Gamma_d &= \Gamma_1\sin^2\theta + \Gamma_2\cos^2\theta, \end{aligned} \tag{19}$$

Note that we have defined defined $\hat{\sigma}_d^- = |d\rangle\langle e|$ and $\hat{\sigma}_b^- = |b\rangle\langle e|$.
*Hint*: For the Hamiltonian part use the results of question 3 of Problem's Set # 3.

(b) Our basic conclusions in the following will not be affected by the simplification $\Gamma_1 = \Gamma_2 = \Gamma$ such that the master equation reads:

$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[\tilde{H}, \hat{\rho}] + \Gamma\left(\mathcal{D}[\hat{\sigma}_b^-]\hat{\rho} + \mathcal{D}[\hat{\sigma}_d^-]\hat{\rho}\right). \tag{20}$$

Note that the excited state $|e\rangle$ decays to both the bright $|b\rangle$ and the dark state $|d\rangle$. However, note that only the state $|b\rangle$ is pumped to the excited state $|e\rangle$ by the external field. Make a schematic figure (as in Fig. 1) with the dark-bright-excited levels at the Raman resonance ($\Delta_g = 0$). Note that the dark state $|d\rangle$ is decoupled from the bright state $|b\rangle$ and thus the steady state should bring all the population to the dark state $|d\rangle$. This surprising effect is **coherent population trapping**.

Let us demonstrate this phenomena. Considering the master equation (20), show that the optical Bloch equations are given by:

$$\begin{aligned}
\dot{\rho}_{ee} &= \mathrm{i}\frac{\Omega_b}{2}(\rho_{eb} - \rho_{be}) - 2\Gamma\rho_{ee} \\
\dot{\rho}_{dd} &= \mathrm{i}\frac{\Omega_g}{2}(\rho_{db} - \rho_{bd}) + \Gamma\rho_{ee} \\
\dot{\rho}_{bb} &= -\mathrm{i}\frac{\Omega_g}{2}(\rho_{db} - \rho_{bd}) - \mathrm{i}\frac{\Omega_b}{2}(\rho_{eb} - \rho_{be}) + \Gamma\rho_{ee} \\
\dot{\rho}_{ed} &= \dot{\rho}_{de}^* = \left(\mathrm{i}\Delta_d - \frac{\Gamma}{2}\right)\rho_{ed} + \mathrm{i}\frac{\Omega_g}{2}\rho_{eb} - \mathrm{i}\frac{\Omega_b}{2}\rho_{bd} \\
\dot{\rho}_{eb} &= \dot{\rho}_{be}^* = \left(\mathrm{i}\Delta_b - \frac{\Gamma}{2}\right)\rho_{eb} + \mathrm{i}\frac{\Omega_g}{2}\rho_{ed} + \mathrm{i}\frac{\Omega_b}{2}(\rho_{ee} - \rho_{bb}) \\
\dot{\rho}_{bd} &= \dot{\rho}_{db}^* = \mathrm{i}(\Delta_d - \Delta_b)\rho_{bd} + \mathrm{i}\frac{\Omega_g}{2}(\rho_{bb} - \rho_{dd}) - \mathrm{i}\frac{\Omega_b}{2}\rho_{ed}
\end{aligned} \tag{21}$$

(c) Show that at the Raman resonance $\Delta_1 = \Delta_2$, the steady state solution $\dot{\rho}_0(t) = 0$, leads to $\langle d|\rho_0(t)|d\rangle = 1$, which demonstrates the coherent population trapping effect.

(d) To see even better the effect plot $\langle e|\rho_0(t)|e\rangle$ as a function of $\Delta_2/\Gamma$ (from $-5$ to $5$) using $\Delta_1 = 0$, $\Gamma_1 = \Gamma_2 = \Gamma$ and $\Omega_1 = \Omega_2 = 0.2\Gamma$.