Problem's set #3

THEORETISCHE QUANTENOPTIK UND QUANTEN INFORMATION
2014S PS 705854
Oriol Romero-Isart
Due Friday 09.05.14

Atom interacting with quantum light

1. **Quantization of an electromagnetic field mode within "macroscopic" classical boundaries: optical cavity mode**. In the first chapter of the theory lecture we described how the electromagnetic field is quantized in free space. In principle, this is all what is needed to describe the interaction of the EM field with matter at the quantum level. However, in some situations it is useful to treat macroscopic boundaries of matter in terms of classical boundary conditions (e.g the mirrors in an optical cavity). In this exercise we will sketch how to quantize the EM field in these situations.

   (a) Using the Maxwell equations for the free field (zero currents and charges), show that in the Coulomb gauge the potential vector $\mathbf{A}$ fulfills the equation

   $$\nabla^2 \mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} = 0 \tag{1}$$

   Hint: Use the identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$, and recall in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and the scalar potential is the Coulomb potential created by the charges.

   (b) Using the separation of variables

   $$\mathbf{A}(\mathbf{r},t) = \alpha(t)\mathbf{f}(\mathbf{r}) + \alpha^*(t)\mathbf{f}^*(\mathbf{r}) = 2\text{Re}\left[\alpha(t)\mathbf{f}(\mathbf{r})\right], \tag{2}$$

   show that $\alpha(t) = e^{-\mathrm{i}\omega t}\alpha(0)$ and

   $$\left(\nabla^2 + k^2\right)\mathbf{f}(\mathbf{r}) = 0, \tag{3}$$

   where $\omega = kc$. Equation (3) is the Helmholtz equation which allows us to obtain the mode function $\mathbf{f}(\mathbf{r})$.

   (c) Show that the transvere electric field can be written as

   $$\mathbf{E}_\perp(\mathbf{r},t) = -\mathrm{i}\omega\alpha(0)e^{-\mathrm{i}\omega t}\mathbf{f}(\mathbf{r}) + \text{H.c.} \tag{4}$$

   (d) Show that the electromagnetic Hamiltonian $H = \epsilon_0 \int d^3\mathbf{r} \left(|\mathbf{E}_\perp|^2 + c^2|\mathbf{B}|^2\right)/2$ can be expressed as

   $$H = \frac{\epsilon_0}{2}\int d^3\mathbf{r}\left(|\mathbf{E}_\perp|^2 + \omega^2|\mathbf{A}|^2\right), \tag{5}$$

   Hint: Use: the identity $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$ with $\mathbf{G} = \nabla \times \mathbf{A}$ and $\mathbf{F} = \mathbf{A}$, the identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$, the Coulomb gauge condition, the fact that $\mathbf{A}$ fulfills the Helmholtz equation, and the divergence theorem (using the fact that the surface integral vanishes of a consequence of the assumed boundary conditions of $\mathbf{A}(\mathbf{r})$.

(e) Show that the quantization

$$\alpha(0) \to \sqrt{\frac{\hbar}{2\omega\epsilon_0}}\hat{a}, \tag{6}$$

where $[\hat{a}, \hat{a}^\dagger] = 1$, leads to $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. Thus the operator describing the electric field observable is given by

$$\hat{\mathbf{E}}_\perp(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\mathbf{f}(\mathbf{r})\hat{a} - \text{H.c.} \tag{7}$$

(f) Now consider a one-dimensional cavity consisting of two mirrors of area A positioned at $z = 0$ and $x = L$ (this is a Fabry-Perot resonator). The boundary conditions are $\mathbf{E}(z = 0) = \mathbf{E}(z = L) = 0$. Show that

$$\mathbf{f}_{n,\epsilon}(z) = \vec{\epsilon}\sqrt{\frac{2}{V}}\sin(k_n z), \tag{8}$$

where $k_n = n\pi/L$, $V = AL$, and $\vec{\epsilon}$ is the polarization vector (perpendicular to $\mathbf{k}$) fulfilling $|\vec{\epsilon}| = 1$ are orthogonal ($\int_V d\mathbf{r}\mathbf{f}_n(z)\mathbf{f}_m(z) = \delta_{nm}$) solutions of the Helmholtz equation (3). Thus the transverse electric field operator can be written as

$$\hat{\mathbf{E}}_\perp(z) = \sum_n\sum_{\vec{\epsilon}} \mathrm{i}\vec{\epsilon}\sqrt{\frac{\hbar\omega_n}{V\epsilon_0}}\sin(k_n z)\hat{a}_{n\epsilon} - \text{H.c.}, \tag{9}$$

where $\omega_n = k_n c$. For a cavity of $L = 1$ cm, what is the value of the free spectral range $\Delta\omega = \omega_{n+1} - \omega_n$? Note that a single mode can be individually addressed, as considered in section 3.1 of the theory lecture.

2. **Spontaneous emission of an atom in confined spaces**. In the theory lecture we have obtained by several methods that the spontaneous emission rate of a two-level isotropic atom in free space is given by:

$$\Gamma_0 = \frac{|\vec{d}_{eg}|^2\omega_{eg}^3}{3\hbar\pi\epsilon_0 c^3}. \tag{10}$$

In this exercise we want to compute the spontaneous emission of a two level atom in confined spaces (*e.g.* in between two infinite parallel conducting planes). To do this, let us start considering a perfectly conducting box of lengths L_x, L_y, and L_z (with one corner at the origin).

(a) Show that under these conditions, the Helmholtz equation (3) has the following orthogonal solutions

$$\mathbf{f}_{\mathbf{k},\epsilon}(\mathbf{r}) = \sqrt{\frac{8}{V}}\begin{pmatrix} \epsilon_x\cos(k_x x)\sin(k_y y)\sin(k_z z) \\ \epsilon_y\sin(k_x x)\cos(k_y y)\sin(k_z z) \\ \epsilon_z\sin(k_x x)\sin(k_y y)\cos(k_z z)\end{pmatrix}, \tag{11}$$

where $V = L_x L_y L_z$, $k_x = n_x\pi/L_x, k_y = n_y\pi/L_y, k_z = n_z\pi/L_z$, and ϵ_i is the i-component ($i = x, y, z$) of the polarization vector $\vec{\epsilon}$ (which is orthogonal to $\mathbf{k}$). Thus, the transverse electric field operator is given by

$$\hat{\mathbf{E}}_\perp(\mathbf{r}) = \sum_{\mathbf{k}}\sum_{\vec{\epsilon}} \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0}}\mathbf{f}_{\mathbf{k},\epsilon}(\mathbf{r})\hat{a}_{\mathbf{k}\epsilon} - \text{H.c.}. \tag{12}$$

(b) Considering that the two level atom is at $\mathbf{r}_0 = (L_x/2, L_y/2, L_z/2)$, show that interaction

Hamiltonian is given by

$$\hat{V} = \hbar \sum_{\mathbf{k}} \sum_{\vec{\epsilon}} \left[g_{\mathbf{k}\epsilon} |e\rangle\langle g| \hat{a}_{\mathbf{k}\epsilon} + \text{H.c.} \right], \tag{13}$$

where

$$g_{\mathbf{k}\epsilon} = \mathrm{i} \frac{\mathbf{f}_{\mathbf{k},\epsilon}(\mathbf{r}) \cdot \vec{d}_{eg}}{\hbar} \sqrt{\frac{\hbar \omega_{\mathbf{k}}}{2\epsilon_0}} \tag{14}$$

What approximations have been done to obtain this Hamiltonian and when are they valid?

(c) Using the resolvent operator we showed in Section 3.4.5 of the theory lecture that the spontaneous emission is approximately given by

$$\Gamma \approx \frac{2\pi}{\hbar} \sum_{\mathbf{k}} \sum_{\vec{\epsilon}} \left| \langle g, \vec{\epsilon}\mathbf{k} | \hat{V} | e, 0 \rangle \right|^2 \delta(\hbar\omega_{eg} - \hbar\omega_{\mathbf{k}}) \tag{15}$$

(this is analogous to the Fermi's golden rule). What approximation are done in order to obtain this expression?

(d) Let us now consider two infinite planes (oriented in the $x-y$ plane) separated by a distance $L \ll \lambda_{eg}/2$, where $\lambda_{eg} = 2\pi c/\omega_{eg}$ is the wavelength associated to the two-level transition. First, discuss why the condition $L \ll \lambda_{eg}/2$ implies that the atom can only couple to the $n_z = 0$ mode, and thus we can just take $k_z = 0$ and sum only over k_x and k_y. Note also than then the possible polarization is fixed to be along the z axis.

(e) Show that for an isotropic atom situated at $\mathbf{r}_0$, one then arrives at

$$\Gamma_{\text{2D}} \approx \Gamma_0 \frac{2(2\pi)^2}{L_x L_y L} \frac{c^3}{\omega_{eg}^3} \sum_{k_x, k_y} \omega_{\mathbf{k}} \sin^2(k_x L_x/2) \sin^2(k_y L_y/2) \delta(\omega_{eg} - \omega_{\mathbf{k}}), \tag{16}$$

where we used the expression of spontaneous emission rate in free space, Eq. (10).

(f) Being infinite planes, we can take the limit $L_x = L_y \to \infty$ and convert the sum into an integral using

$$\sum_{k_x, k_y} \to \frac{L_x L_y}{(2\pi)^2} \int dk_x dk_y, \tag{17}$$

and approximate $\sin^2(k_x L_x/2) \sin^2(k_y L_y/2)$ to its average value $1/4$. Show that then

$$\Gamma_{\text{2D}} \approx \Gamma_0 \frac{\pi c}{\omega_{eg} L} = \Gamma_0 \frac{\lambda_{eg}}{2L}. \tag{18}$$

Is Γ_{2D} larger or smaller than Γ_0? Note that $\Gamma_{\text{2D}} \propto \omega_{eg}^2$.

3. **Collapse and Revival in the Jaynes-Cummings model**. Consider a two level system interacting with a single electromagnetic mode. Its Hamiltonian in the rotating wave approximation and assuming a real Rabi frequency is given by the Jaynes-Cummings Hamiltonian

$$\hat{H} = -\hbar\Delta |e\rangle\langle e| + \frac{\Omega}{2} \left(\hat{a}\hat{\sigma}^+ + \hat{a}^\dagger \hat{\sigma}^- \right). \tag{19}$$

(a) Show that the Rabi oscillations for the state $|g\rangle \otimes |n\rangle$ are given by

$$P_n(t) \equiv \left| \langle e| \otimes \langle n-1| e^{-\mathrm{i}\hat{H}t/\hbar} |g\rangle \otimes |n\rangle \right|^2 = \frac{1}{2} \frac{\Omega_n^2}{\Omega_n^2 + \Delta^2} \left[1 - \cos\left(t\sqrt{\Omega_n^2 + \Delta^2} \right) \right], \tag{20}$$

where $\Omega_n = \sqrt{n}\Omega$.

(b) Consider now that the initial state is given by $|g\rangle \otimes |\alpha\rangle$ where $|\alpha\rangle$ is a coherent state of the electromagnetic mode. Show that the probability to measure the two level system in the state $|e\rangle$ is given by

$$P_e(t) = \sum_{n=1}^{\infty} p_n P_n(t), \tag{21}$$

where

$$p_n = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}. \tag{22}$$

(c) Considering $\Delta = 0$, make a plot of $P_e(t)$ as a function of Ωt for different α. Observe the collapse of the Rabi oscillations and the revival starting at roughly $t_R\Omega \sim 2\pi|\alpha|$. Note that the larger $|\alpha| = \bar{n}$, the better is the collapse and revival observed.

4. **A relation in complex analysis useful for quantum optics**. In quantum optics, we often find the following integral

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx, \tag{23}$$

where $f(x)$ is a continuous function.

(a) Assuming that $f(z)$, for $z \in \mathbb{C}$, fulfills $|f(z)| \to 0$ as $|z| \to \infty$ and is analytic in the real axis, show that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x-a \pm \mathrm{i}\epsilon} dx = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx \mp \mathrm{i}\pi f(a), \tag{24}$$

where

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{a-\epsilon} \frac{f(x)}{x-a} dx + \int_{a+\epsilon}^{\infty} \frac{f(x)}{x-a} dx \right] \tag{25}$$

is the Cauchy principal value of the improper integral. One typically writes these equalities as

$$\lim_{\epsilon \to 0} \frac{1}{x-a \pm \mathrm{i}\epsilon} = \mathcal{P}\frac{1}{x-a} \mp \mathrm{i}\pi\delta(x-a), \tag{26}$$

which make sense only when considered as integrands.

(b) Use the previous result to show that

$$\int_0^{\infty} d\omega f(\omega) \int_0^{\infty} dt e^{-\mathrm{i}(\omega-\omega_0)t} = f(\omega_0)\pi - \mathrm{i}\mathcal{P} \int_0^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0} \tag{27}$$

(we assume $\omega_0 \gg 0$) which appears very often in quantum optics. The imaginary term containing the Cauchy principal value is typically related to level shifts. For instance, we omitted (on purpose) this more careful integration which would lead to a Lamb shift in the theory lecture (Sec. 3.2.) when discussing spontaneous emission with the Wigner-Weisskopft approximation.

5. **Adiabatic elimination in the Lambda scheme: Raman transitions**. In a Lambda scheme (recall problem 3 in Problem's set # 2), the total Hamiltonian in the rotating frame and after performing the rotating wave approximation can be written as $\tilde{H} = \hat{H}_0 + \hat{V}$, with

$$\hat{H}_0 = \hbar\Delta_1|g_1\rangle\langle g_1| + \hbar\Delta_2|g_2\rangle\langle g_2|, \tag{28}$$

$$\hat{V} = \frac{\hbar\Omega_1}{2}\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \frac{\hbar\Omega_2}{2}\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right). \tag{29}$$

We have defined $\Delta_j = \omega_j - \omega_{g_j}$, and the Rabi frequencies which are assumed to be real $\Omega_j = -2\vec{d}_{ej} \cdot \vec{\epsilon}_j E_j/\hbar$ (the laser phases are chosen appropriately). Assuming the drivings are

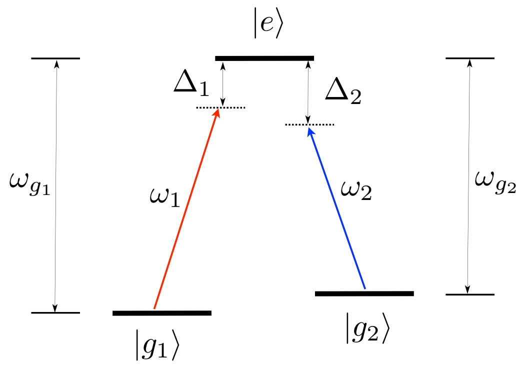

Figure 1: Lambda scheme.

off-resonant, we want to adiabatically eliminate the level $|e\rangle$, namely, we want to derive an effective Hamiltonian for the levels $\{|g_1\rangle, |g_2\rangle\}$.

(a) Using the Schrieffer-Wolf transformation which is defined by a unitary operator $\hat{U} = \exp[\mathrm{i}\hat{G}]$ with $\hat{G}$ being an Hermitian block-off-diagonal operator. We want to obtain the effective Hamiltonian in second order perturbation theory in $\hat{V}$. Thus, it is sufficient to obtain $\hat{G}$ in first order in $\hat{V}$. Show that this is given by:

$$\hat{G} = \frac{\mathrm{i}\Omega_1}{2\Delta_1}\left(|g_1\rangle\langle e| - |e\rangle\langle g_1|\right) + \frac{\mathrm{i}\Omega_2}{2\Delta_2}\left(|g_2\rangle\langle e| - |e\rangle\langle g_2|\right) + \mathcal{O}\left(V^2\right) \tag{30}$$

(b) Using the previous result, show that the effective Hamiltonian $\hat{H}_{\text{eff}}$ of the subspace $\{|g_1\rangle, |g_2\rangle\}$ is given by

$$\hat{H}_{\text{eff}} = \hbar\tilde{\Delta}_1|g_1\rangle\langle g_1| + \hbar\tilde{\Delta}_2|g_2\rangle\langle g_2| + \hbar g\left(|g_1\rangle\langle g_2| + |g_2\rangle\langle g_1|\right) + \mathcal{O}\left(V^3\right), \tag{31}$$

where the level shifts and the induced coupling strength are given by

$$\begin{aligned}\tilde{\Delta}_{1(2)} &= \Delta_{1(2)} + \frac{\Omega_{1(2)}^2}{4\Delta_{1(2)}},\\ g &= \frac{\Omega_1\Omega_2}{8}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right).\end{aligned} \tag{32}$$

Note that by adiabatically eliminating the excited state, now the two ground states are effectively coupled. Indeed, this Hamiltonian looks exactly the same as the one obtained by a two level atom interacting with a nearly-resonant "classical" field. The transitions between the levels $|g_1\rangle$ and $|g_2\rangle$ induced by this effective coupling (via the excited state) are called **Raman transitions**.